Modeling White Dwarf Star Magnetization

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For our research, we derived an equation of state for the white dwarf star from exterior to interior. We used Chandrasekhar Equation along with observational data for Sirius B. The results obtained from our equations predict the mass of Sirius B to be 2×10^{30}kg, the internal temperature to be 3×10^{7}K, and the central density to be 2.45×10^{10}g/cm^3. We then investigated the possibility of spontaneous magnetization following the work of Akheizer. Our results predict a maximum internal magnetization of the order of 10^{13}A/m. This result is in agreement with previous prediction with different approach.

I. INTRODUCTION

The evolutionary of stars generally starts with the main sequence stars. Main sequence stars go through nuclear fusion of hydrogen at the core to generate high pressure that can prevent the stars from collapsing by the gravity. When the nuclear fusion ends, main sequence stars with small mass proceed to a giant phase which fuses next level of the elements. When stars cannot continue their nuclear fusions, their outer layers explode. After the explosion, stars become very dense and hot, and the remnants of the giant stars are called white dwarf stars.

A white dwarf star is a remnant of a red giant star which went through the hydrogen fusion and helium fusion process. Hence in the composition of stars, we assume there will be almost no presence of Hydrogen and Helium, and most of the elements will be Carbon, Nitrogen, Oxygen, and some elements that are heavier. A white dwarf star starts of with very high temperature but for billions of years, it cools down. Because a white dwarf starts with small radius, its cooling process takes tremendous amount of time. The cooling process comes from radiating the heat, and because there is no more fusion, or producing heat energy, the cooling process slows down gradually. A white dwarf star is believed white due to its color of the temperature on H-R diagram. However, most of the white dwarf stars have the surface temperature that is still hotter than that of the Sun , and as dense as 1.44 solar mass when the radius is only about the Earth's radius. The mass and radius of a white dwarf star is found to be in inversely proportional relation, in another words, if the star is more massive, then the radius will be smaller. In other words, the star will become more dense. However, the mass is limited up to 1.4 solar mass by Chandrasekhar' limit, which explains that after 1.4 solar mass, the gravitational force will dominate the pressure generated by relativistic, degenerate electrons inside the white dwarf star, and the star will collapse.

The true mystery about these white stars is that many of white dwarf stars are founded to have magnetic fields around them, and some white dwarf stars have over 1 million Gauss, or 100 Tesla. The source of the magnetization can be either external or the spontaneous magnetization, but the accurate reason for the strong magnetization on white dwarf stars are still unexplained. However, we assume that some of these magnetization can be from spontaneous magnetization under certain conditions.

For many years, scientists predicted magnetization of a white dwarf with different approaches. Some scientists predicted a white dwarf to be in a LOFER(Landau Orbital Ferromagnetic) state, or the electrons are in a balance state. Hyung Joon Lee, in his work, predicted the magnetization to be 10^7G[1], and another study predicted 10^{13}G[2] also under assumption of a white dwarf being in a LOFER state. However, other studies such as one by R.F.O' Connell and K.M. Roussel[3] predicted that only certain white dwarfs can be accounted for the presence of magnetic fields because a LOFER state requires thermal equilibrium.

If we assume that magnetization on a white dwarf star comes from spontaneous magnetization, we have to consider what forms spontaneous magnetization. Spontaneous magnetization comes from an alignment of the fermions, electrons with other ions. Stern-Gerlach experiment proved that electrons have the intrinsic magnetic moments, and are capable of making spontaneous magnetization due to an alignment of the intrinsic magnetic moments. The alignment happens in the quantum energy state called degeneracy. Fermions have ½ intrinsic spin and when fermions become degenerate, according to their energy state, they move into each of the energy state. However, Pauli Exclusion Principle limits that two electrons with the same set of quantum numbers cannot exist next to each other, and multiple fermion state wave functions have to be anti-symmetric, and because of the anti-symmetric property if the space becomes anti-symmetric, the spins of fermions become symmetric and aligned. Due to the intrinsic magnetic moments in fermions, the alignment of spin generates magnetization. However, the spatial anti-symmetric that causes the magnetization comes from the relativistic-degenerate electrons.

As we know that a white dwarf star is made up of certain elements like Helium, Carbon, Nitrogen, Oxygen and so on. Thus, we also know that the star has to electrons from these elements, and these electrons can be separated into to phases: non-relativistic and relativistic. Non-relativistic meaning the electrons are moving much less than speed of light, and relativistic meaning the electrons are moving close to the speed of light. However, because of the hydrostatic equilibrium presenting in a white dwarf star, we have to assume that a white dwarf star is relativistic.

Sirius B, one of the well known white dwarf stars, is an interesting star. Its mass is known to be close to one solar mass, and the radius is about Earth's radius. We assumed the star to be relativistic and degenerate because of its high density and pressure. With its accurately calculated radius and mass, Sirius B is our perfect example to create a magnetization model of white dwarf stars. In the next sections, we will discuss the important theories to build the model for magnetization of Sirius B that shows how magnetization varies with the radius of the star using the models for mass, density, and temperature of the star approaching from the exterior to interior of the star.

II. Theory

The relativistic movement of the electrons in Sirius B can be proved by the hydrostatic equilibrium of the star. According to Newton's gravitational law, every object with mass has to be affected by the gravity. That is if the star is massive, the gravity is stronger, and the star has to be collapsed by the gravity if there is no other opposite fore that is against it. Since Sirius B is a white dwarf star, the hydrogen fusion at the core has been stopped which makes the star unprotected from the center gravity. However, as we can observe, the star is untouched and stays in an equilibrium. This equilibrium is called hydrostatic equilibrium, and it forms by the pressure generated by the relativistic movement of the electrons. Therefore Sirius B has to be relativistic.

The hydrostatic equilibrium then leads to a dimensionless equation, Lane Emden Equation[4]. We need a dimensionless system to avoid complexity with the units in the equation. Lane Emden Equation assumes a star to be a completely degenerated star, and it gives a solution as pressure and density related to the radius by a polytrope equation. Polytrope equation can be applied to two different phases, non-relativistic degenerate and ultra-relativistic degenerate, and the gamma changes according to the phase by 5/3 for non-relativistic degenerate state, and 4/3 for ultra-relativistic degenerate state. However, the polytrope equation fails to be applied on a white dwarf star, because the equation has a condition that the radius has to be infinity. By the Chandrasekhar's limit and other observations, we know that white dwarf stars do not have infinite radius. Polytrope equation is commonly used for gases not for solid objects like a white dwarf star. Thus, although Lame Emden Equation solves for the relation of the density and mass relatively to the radius of a star, however, the solution by a polytrope equation cannot be applied. Fortunately, there is another method to solve Lame Emden Equation, the Chandrasekhar's equation[5].

Chandrasekhar's equation comes from Lame Emden equation as well. The difference from the polytrope equation is that Chandrasekhar uses different symbols for ξ, and the equation actually works for a solid object. Even though, the equation requires many substitutions to solve, once we convert the equation into dimensionless method using dimensionless unit substitution,
Chandrasekhar's equation gives the density model for a white dwarf star. From the density equation we can derive our final model for mass of a white dwarf star.

To obtain actual calculated data, we will use Fortran. However, our final models for density and mass uses derivatives, and Fortran does not support calculation of differential equation. Fortunately, we can use the Euler's method to change the differential into differences. Thus, the differential equations become multiple loops of calculation. With all the data points from Fortran calculation, we will get accurate graphs of density vs. radius and mass vs. radius. And from those density and mass functions, we will get the temperature as a function of radius.

Magnetization of a star comes from the intrinsic magnetic moments of the electrons inside the star. These magnetic moments are described in two states: spin up and spin down. When these magnetic moments are aligned, magnetization forms, and to be exact, a spontaneous magnetization is formed. According to A.I. Akheizer, whether the magnetic moments are spin up or spin down, magnetization can be expressed as the difference of the two Fermi-Dirac integrals. Expanding his theory and dimensionless units, we should be able to express the magnetization in terms of temperature.

III. COMPUTATION

$$F_{e^-}+F_g=0 \quad (1)$$

The derivation of the density model comes from the hydrostatic equilibrium equation (1). If we let mass (M) as a function of radius(r), the mass becomes the integration from 0 to the radius of the surface area multiplied by the density, and we can solve the mass at any r. Thus, we have the mass equation (2).

$$M(r)=\int_0^r \rho(r')4\pi r^2 dr' \quad , (2)$$

If we evaluate the hydrostatic equilibrium using the mass equation, we get

$$\frac{dP_{e^-}}{dr}dV=\frac{-GM(r)dM}{r^2} \quad , (3)$$

If we substitute the M(r) with the integration, and then evaluate the equation further we get

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}dP_{e^-})=-4\pi G\rho \quad , (4)$$

This equation is the base of Lame Emden Equation. However, we need to convert it to dimensionless using dimensionless method. For dimensionless method we will use substitutions as follow:

$$\mathrm{P}=\mathrm{K}\rho^\gamma \quad (5)$$

$$\rho=\rho_c\theta^n \quad (6)$$

$$r=\xi a \quad (7)$$

For γ, we will assume our model, Sirius B, is in ultra-relativistic regime, thus we will use 4/3. For ρ_c, we will assume that the critical density is around 2.45×10^{10}g/cm^3. To solve a , for r, we will use

$$a=\sqrt{\frac{(n+1)P_c}{4\pi G\rho_c}} \quad , (8)$$

where we will assume P_c, critical pressure, is about 10^{17} atm.

If we substitute these equations into (3), we will get Lame Emden Equation (9)

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})=-\theta^n \quad . (9)$$

To solve for the Lame Emden equation, we will use Chandrasekhar's equation of state to re-express equation (9) .

$$\frac{1}{\eta^2}\frac{d}{d\eta}(\eta^2\frac{d\phi}{d\eta})=-(\phi^2-\frac{1}{y_0^2})^{\frac{3}{2}} \quad (10)$$

Using the substitution for the dimensionless method , the Chandrasekhar's equation becomes dimensionless. For substitution, we use

$$\eta=\eta_1\xi \quad . (11)$$

The Chandrasekhar's equation becomes

$$\frac{d^2\phi}{d\xi^2}+\frac{2}{\xi}\frac{d\phi}{d\xi}=-\eta^2(\phi^2-\frac{1}{y_0^2})^{\frac{3}{2}} \quad . (12)$$

where y_0, we use

$$y_0=((\frac{\rho_c}{\beta})^{\frac{2}{3}}+1)^{\frac{1}{2}} \quad . (13)$$

In the equation (2), we will use Chandrasekhar's equation for density (14)

$$\rho=\rho_c\frac{y_0^3}{y_0^2-1^{3/2}}(\phi^2-\frac{1}{y_0^2})^{\frac{3}{2}} \quad . (14)$$

where ρ_c is

$$\rho_c=\beta\xi_0^3 \quad , (15)$$

where β is

$$\beta=\frac{8\pi m^2c^3\mu_e m_H}{3\mathrm{h}^3} \quad , (16)$$

m=mass of the star, c=speed of light, μ_e=Bhor Magneton,

m_H=mass of hydrogen atom, h=Plank constant

where y_0^2 is

$$y_0^2=\xi_0^2+1 \quad . (17)$$

For r, in equation (2)

$$r=\eta_1\alpha \quad , (18)$$

where α is

$$\alpha=(\frac{2\mathrm{A}}{\pi G})^{\frac{1}{2}}\frac{1}{\beta}((\frac{\rho_c}{\beta})^{\frac{2}{3}}+1)^{-\frac{1}{2}} \quad , (19)$$

ρ_c=Critical density, G= Gravitational constant

where A is

$$A=\frac{\pi m^4c^5}{3\mathrm{h}^3} \quad . (20)$$

Thus equation (2) becomes our equation for mass as a function of radius.

$$M=-4\pi\beta((\frac{\rho_c}{\beta})^{\frac{2}{3}}+1)^{\frac{3}{2}}\alpha^2\eta_1^2\frac{d\phi}{d\xi}|\xi=1 \quad . (21)$$

Now we have mass and density as functions of radius. These will help to define temperature as a function of radius. Temperature can be expressed as a function of Luminosity. Since we know that temperature as a function of radius can be expressed as

$$\frac{dT}{dr}=\frac{-3\mathrm{K}_c\rho L}{16\pi acR^2T^3} \quad . (22)$$

The luminosity of star is proportional to the mass of the star, thus L becomes

$$\frac{dL}{dr}=\epsilon\frac{dM}{dr}$$, (23)

where ε is 0.026 for Sirius B, and we can replace ρ with equation (14), $\frac{dL}{dr}$ with equation (21). Thus, the luminosity is now expressed in terms of mass and radius.

We will assume a white dwarf star cools itself through only conduction. Therefore, we will consider only the conduction opacity, K_c which tells us how well the energy can transfer inside the star. K_c is expressed as

$$K_c=\frac{16\sigma T^3}{3\rho\chi}$$ [6] , (24)

where σ is a constant, 5.67x10^{-8}, and χ is expressed as

$$\chi=\frac{a\rho K_B^2 T}{m_e m_H \nu_{ee}}$$, (25)

where a= $\frac{\pi^2}{3}$, K_B is Boltzmann-constant, and υ_{ee} is the electron-electron scattering.

We investigated the temperature dependence of electron-electron scattering and found the variation of temperature was minimum. Therefore, we assume υ_{ee} to be 1.6x10^{18}.

With all the substitutions, we have the final model for our temperature as a function of radius. Equation (22) becomes

$$T=\frac{3m_e m_H M}{4\pi^2 K_B^2 R^2\rho T}\epsilon\frac{L\odot}{M\odot}$$. (26)

Now we have expressed mass, density, and temperature in terms of radius. With theses equations (6,7,9), we can develop a magnetization as a function of radius using the fundamental equation of magnetization

$$M=\mu(n_\uparrow-n_\downarrow)$$, (27)

where μ is magnetic moment, and $\mu(n_\uparrow-n_\downarrow)$ is the difference between spin up and spin down states of electrons.

Using Akheizer's general approach[7] including his definition of dimensionless units to investigate the magnetization of stellar interiors, we have extended his work into the ultra-relativistic regime which produces the following equation for the magnetization

$$\chi=\tau^3[\psi(z_+)-\psi(z_-)]$$, (28)

the Fermi-Dirac integral, $\psi(z)=\int_0^\infty\frac{x^n dx}{\exp(x-z)+1}$ (29), because we assume Sirius B to be in an ultra-relativistic regime, we will say n=2, and the integral becomes a finite expansion when T→0. Then, $\psi(z)$ becomes

$$\psi(z)=\frac{1}{3}(z^3+\pi^2 z)$$ [8]. (30)

For equation (11), χ can be also calculated as

$$\chi=\frac{8\alpha^2\nu^3\mu_B M}{9\pi_e^{2m}c^2}$$, (31)

where μ_B is Bohr Magneton of the electron, $\alpha=\frac{e^2}{\hbar c}$ (32), and $\nu=\frac{1}{\mu_0(\chi+1)}\frac{\gamma_2}{\mu_B^2}$ (33) where χ is the average magnetic susceptibility of carbon, and we will use -2x10^5 .

Also z(±) = $\frac{\beta\pm x}{\tau}$ (34), where

$$\beta=\frac{8\alpha^3\nu^3\lambda_e^3}{9\pi m_e c^2}\mu'$$ (35) and $\mu'=\mu-\gamma_1 Ne$ (36)

where $\mu=\epsilon_F\left(1-\frac{\pi^2}{3}\left(\frac{K_B T}{\epsilon_F}\right)^2\right)$. (37)

γ_1 and γ_2 are related to the repulsive and attractive components of the potential energy.

Between electrons, where γ_1 is the potential energy of Coulombic Repulsion, and γ_2 is the potential energy of spin-spin interaction. γ_1 can be expressed as γ_1 = (1.21x10^{-11})^{3}Ec, and γ_2 can be written as γ_2 = (1.21x10^{-11})^{3}Ess. 1.21x10^{-11} is the assumed mean free path which we will vary later. We will let Ec < Ess < 10Ec to see how magnetization varies by the change of the potential energy of spin-spin interaction. We will also let Ess to be always greater then Ec, and this is proved by the work of E.E Salpeter[9].

τ can be expressed as a function of temperature, and it becomes $\tau=\frac{8\alpha^2\nu^2}{9\pi m_e c^2}T$. (38)

When we substitute z with equation (34), now we have $2\chi^2=1+6\beta^2-2\tau^2$. (39)

Transforming equation (39) back to our non-unitless system, we have magnetization as a function of radius which is equation (40).

$$M=\frac{\mu_B}{\gamma_2}\sqrt{\frac{1}{2}\left(\frac{9\pi^4 m_e c^2\mu_B^4}{8\alpha^2\gamma_2^2}\right)^2+6\mu'^2-2K_B T^2}$$

(40)

IV. RESULTS AND ANALYSIS

Fig 1. Density vs. Radius

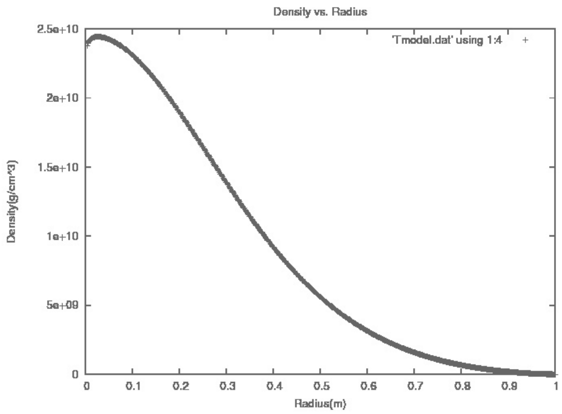


Fig 2. Mass vs. Radius

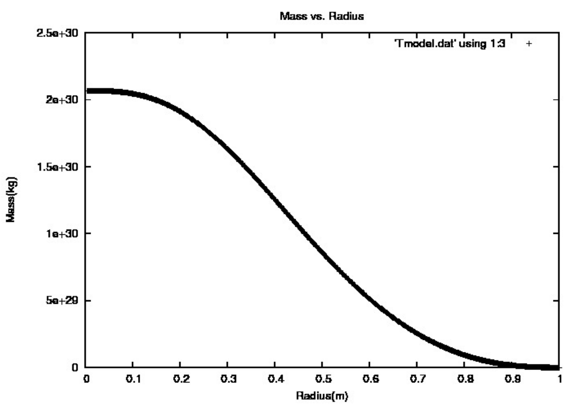
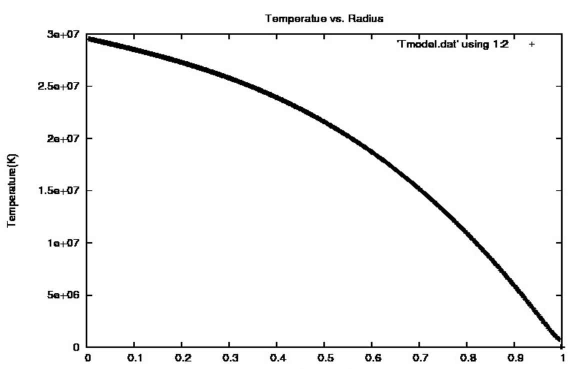


Fig 3. Temperature vs. Radius



The result of our temperature model of Sirius B shows that at the core, the temperature is at the highest, $3x10^7$K. Most people predicts the core of Sirius B to be around 10^7K to 10^8K. Hence, our calculation of the temperature of Sirius B is reasonable. As we go out to the surface of the temperature, we can see that the temperature dramatically drops down after we pass half of the radius. However, the temperature never reaches 0K, or absolute zero, and it is understandable that absolute zero is unachievable by a star. At the core, the graph also proves that Sirius B is not in an isothermal state, which also proves that previous theory of a white star being non-degenerate is not accurate since non-degenerate means isothermal at the core. Clearly, temperature is varying as the radius is changing, but the variation of temperature at the core is not big enough. As we assumed Sirius B to be in ultra-relativistic regime, we expected the temperature variation at the core to be high, but the result opens to a possibility that Sirius B could be in rigorous-relativistic regime.

Fig 2. Magnetization in variation of Couloumbic Potential Energy (see last page)
Fig 3. Magnetization in variation of Spin-Spin Interaction Distance (see lastpage)

The result for magnetization came out to be order of 10^{13}A/m. Other approximations by different approaches also predict magnetization of Sirius B to be around 10^{13}A/m. As The above graphs shows, at the surface of Sirius B, there is no magnetization, and we can assume that the surface of Sirius B is in non-degenerate regime. The graphs also show the variation in magnetization by changing Ess, the potential energy of spin-spin interaction. The result is very interesting. As we increase the Ess, the graph shows that the strength of the magnetization is oscillating. There is a possibility that magnetization can only go up to a certain maximum length. Also the second graph show the variation in spin-spin interaction distance. For this calculation we set the Ess at the maximum value, 10 Ess. The result shows almost the same pattern as the result of variation in Ess. As the interaction distance increased the strength of magnetization oscillated. The oscillation of magnetization opens to further study on magnetization of a white dwarf star.

V. CONCLUSION

Starting with observational data for Sirius B, we were able to numerically arrive at an equation of state for the white dwarf star using Chandrasekhar's equation. This equation of state describes how the temperature and density vary with radius. Our solution method differed from other approaches in that it was calculated from exterior to interior of a star. From our solution we predict an interior density of $2.45x10^{10}g/cm^3$ which is in line with previous results [4] and an internal temperature of $3x10^7$K which is also in agreement with other studies[6].

Most notably, we have discovered that the assumption of the white dwarf star being isothermal is not exactly accurate. Our model predict that the temperature drops by a factor of 2 roughly half was from the center and by a factor of 6 as 50% of the radius.

Using our equation of state, we extended the work of Akheizer into the ultra-relativistic regime to develop a model of the magnetization as a function of radius. In our model, we varied the spin-spin interaction term and the interaction distance. We discovered that there is a peak magnetization at a value of Ess=4Ec where Ec is the Coulumbic interaction potential energy and where the electrons are closest to the each other at a value of $\gamma_2 = 3x10^{-9}$. These results all predict internal magnetization values on the order of 10^{13}A/m which is in agreement with other research[2].

VI. REFERENCES

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[12]http://www.cs.mtu.edu/~shene/COURSES/cs201/NOTES/fortran.html

Fig 2 Magnetization in variation of Couloumbic Potential Energy

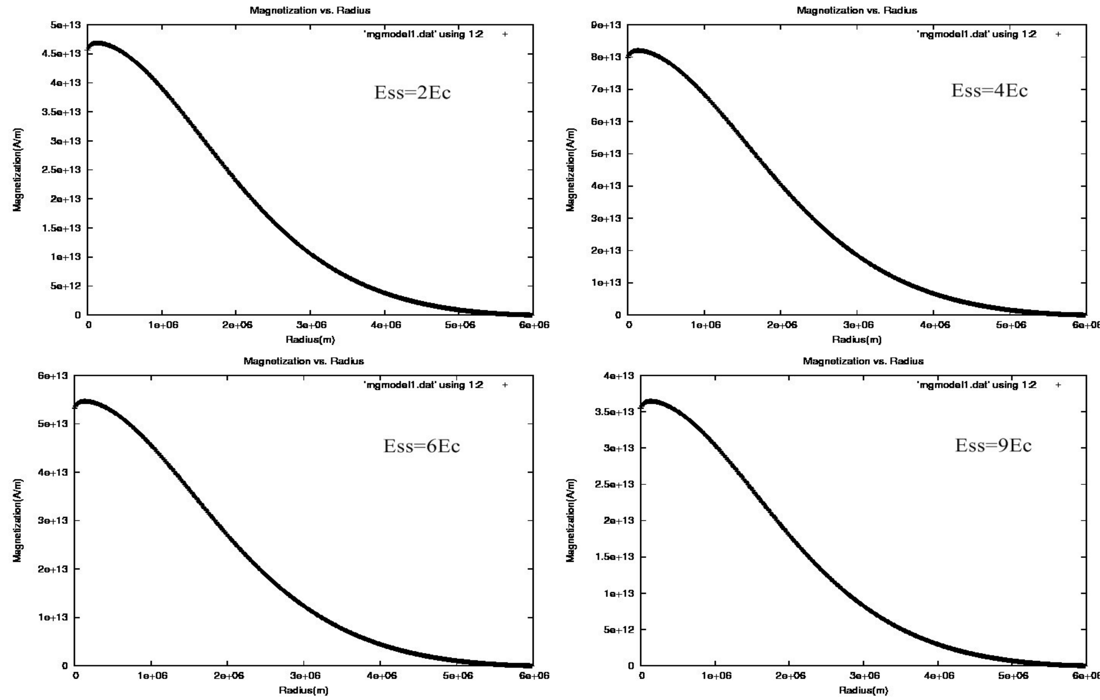

Fig 3. Magnetization in variation of Spin-Spin Interaction Distance