A Simple Planetary Evolution Model Using the Solar Nebular Theory

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A computational and theoretical observation and explanation is obtained for the relative positions of the planets with respect to the sun using a derivation of the wave equation assuming a shock wave propagates throughout the proto-planetary disc from the center of the sun. In order to obtain the analytical solutions, the wave equation is solved in polar coordinates using Bessel's function and trigonometric expressions assuming a constant density throughout the proto-planetary disc. The solutions of the wave equation are plotted, such that pressure is dependent upon radius and it is theta independent, and the nodal points of the plots are observed. The nodal points are theorized to be the locations where the planets should form. The effects of a varying distance on pressure are observed along with a varying constant of the wave equation and varying time. Finally, the effects of a varying density throughout the proto-planetary disc are briefly discussed.

INTRODUCTION

The origin of our solar system is one of the most controversial and relevant topics debated about in the scientific community today. Many theories have been constructed to explain how the sun formed and why the planets are situated as they are now and many of these theories are not adequate in explaining those questions. One rejected theory is that the planets were born-ready from the sun, and the sun ejected them out (Edgeworth 1949). This theory is wrong because any bodies ejected from the sun could not possibly contain the sufficient angular momentum to establish orbits around the sun, and the sun is not capable of ejecting large masses. Other rejected theories include collision theories which suggest that two stars collided with each other and our solar system was the end result of the collision (1949). This theory cannot be true because the angular momentum from the collision would not be sufficient enough to sustain orbits around a star. The only theory that seems to work for our solar system is the Core Accretion Model.

The Core Accretion Model is supported by the Solar Nebula Hypothesis, which explains that the solar system developed after a contraction of a molecular cloud in the Milky Way Galaxy. The theory starts with a dust cloud that starts to collapse due to a collision with another cloud or the impact of a shock wave from an exploding star nearby. The cloud begins to contract as gravity tries to collapse the cloud and pressure tries to expand the cloud. The density of the dust cloud starts to increase until dense cores start to form in the cloud. These dense cores contract further into proto-stars and those proto-stars increase in temperature and density until nuclear fusion starts to occur, resulting in a star (Jones 1999). While the proto-star is increasing its temperature and mass due to accretion, there is a disc of dust already present around the proto-star from the initial contraction of the dust cloud. As the proto-star gains mass, the dust in the disc begins to coalesce and become several fractions of a planet, planetesimals. The dust in and around the planetesimals continues to coagulate until the planetesimals gain enough mass to attract other planetesimals through gravity. As the gravitational attraction increases among planetesimals, the collision rate increases as well and so larger planetesimals collide with smaller ones resulting in the growth of larger planetesimals at the expense of the smaller ones. These planetesimals continue to accrete mass until larger bodies, embryos, form. These embryos form at different distances from the sun and they continue to bombard other embryos until planets form (1999).

This project takes a closer look at the different distances of the planets from the sun and how the Titius-Bode Law predicts those distances. In 1772, the Titius-Bode Law was reiterated by twenty five year old astronomer, Johann Elert Bode, explaining a relationship between the distances of most of the planets from the sun using one equation (Alfven 1908). Many papers theorize an initial propagating shock wave that is swept through the proto-planetary disc which resulted in a pressure build-up at certain points away from the sun (1908). The goal of this work is to find a connection between the propagating shock wave creating a build-up of pressure at nodal points and the Titius-Bode Law using solutions of the Wave Equation. The distances of each nodal point from the initial radius, found by plotting the Wave Equation solutions, should be equal to the planetary distances predicted by the Titius-Bode Law. The solutions of the Wave Equation consist of a Bessel function that varies due to its "n-th" order value. Also, the pressure is assumed to be only dependent upon the varying distances from the initial radius because density is held constant throughout the disc and pressure is angle independent. After some plots can be generated with consistent results, the effects of a varying density throughout the disc are discussed.

THEORY

Based on observations of the solar system, the current size of the solar system is about 100 AU across, the mass is close to 2e+30 kilograms, and the angular momentum of the solar system is 3.2e+43 $\mathrm{kg} * \mathrm{m}^2 * \mathrm{s}^{-1}$ (Kopal 1973). The sun contains about 99.8% of the solar system's mass; however, it only contains 2 % of the total angular momentum of the solar system. Interestingly though, during the proto-planetary stage, the proto-star contained 98% of the angular momentum because of its rapid spin rate and mass (1973). This means that sometime during the proto-planetary stage, the angular momentum was transferred from the proto-sun to the planetesimals or embryos. One theory suggests that pressure waves travel through the medium transferring the angular momentum; sometimes, this process is referred to as turbulence (Jones 1999). The waves are generated due to the proto-sun's rapid spin and they travel throughout the solar system. Anand explains in his research that propagating shock waves tend to converge and concentrate at nodal points away from the source of the shock waves (2012). This theory can be applied to the propagating pressure waves generated by the proto-sun. The pressure waves also concentrate at nodal points from the sun and this research predicts that where the waves are concentrating is where the planets evolved.

METHODS

The research began with the general wave equation:

$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$
$$z = F(x,y)T(t)$$

In this case the z variable in the equation represents the pressure of the propagating wave. I predicted that the pressure is dependent on the distance of the wave from the sun (spatial function, F) and on time (time function, T). The velocity variable in the equation is equal to the speed of sound in air (350 m/s). To solve this equation, I used partial differentiation and the separation of variables method. Using those methods, I simplified the equation down to:

$$T\nabla^2 F = \frac{1}{v^2}F\left(\frac{d^2T}{dt^2}\right)$$

which can now be equated to a constant K:

$$\frac{1}{F}\nabla^2 F = -K^2$$
$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -K^2$$

If we set both equations to equal zero we can rewrite the equations as:

$$\nabla^2 F + FK^2 = 0$$
$$\frac{d^2T}{dt^2} + K^2v^2T = 0$$

The function F is a spatial function that is dependent of the distance (R) of the pressure wave from the sun and the angle (Θ). Therefore the first solution can be simplified further:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \Theta^2} + K^2F = 0$$
$$F = R(r)\Theta(\theta)$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)\Theta(\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R(r)\Theta(\theta)}{\partial \Theta^2} + K^2F = 0$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(\Theta(\theta)\frac{rdR(r)}{dr}\right) + \frac{1}{r^2}R(r)\frac{d^2\Theta(\theta)}{d\Theta^2} + K^2R(r)\Theta(\theta) = 0$$
$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\Theta^2} + K^2r^2 = 0$$

From here I can reapply the separation of variables method to simplify further just for theta since the second term is a function of theta only:

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\Theta^2} = -n^2$$
$$\Theta = \begin{cases}\sin n\theta \\ \cos n\theta\end{cases}$$

This solves for theta. To solve for the distance function (R) I use my new constant, n, in place of the term dependent of theta:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 + K^2r^2 = 0$$
$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (K^2r^2 - n^2)R = 0$$

This equation can be simplified to a simple Bessel function equation such that:

$$R(r) = J_n(Kr)$$

The final function to solve for is the time function (T) and that can easily be done by using similar methods of separation or variables and implementing trigonometric functions:

$$\frac{d^2T}{dt^2} + K^2v^2T = 0$$

$$T = \begin{cases}\sin Kvt \\ \cos Kvt\end{cases}$$

Before the exact solution of pressure is established we must set some boundaries. One boundary is that when time is zero, the pressure is at its maximum because the proto-sun is just beginning to propagate pressure. Another boundary is the final radius of the solar system, which will be represented by the letter, a. With these boundaries we can solve the wave equation down to:

$$z = J_n\left({kr}/{a}\right)\begin{Bmatrix}\sin n\theta \\ \cos n\theta\end{Bmatrix}\begin{Bmatrix}\sin {kvt}/{a} \\ \cos {kvt}/{a}\end{Bmatrix}$$

To use the Bessel function we must solve for k using the equation:

$$v_{mn} = \frac{k_{mn} v}{2\pi a}$$

Where v subscript mn is the speed of sound in hydrogen gas since hydrogen is the most abundant gas in the solar system. The subscript n can only be a positive, even integer throughout the equation and the subscript m does not apply because for my data collection, I did not change the density of the gas. And finally the regular v is the speed of sound in air. Therefore:

$$k = \frac{2\pi a(v_{mn})}{v}$$

Using this fact and the fact that if I sub in zero for time in the sin(kvt/a), the entire equation equals zero which is not possible because I know that pressure should be at maximum when time equals zero; therefore the equation can be simplified further:

$$z = J_n\left({}^{kr}/_{a}\right)\begin{Bmatrix}\sin n\theta \\ \cos n\theta\end{Bmatrix}\left\{\cos {}^{kvt}/_{a}\right\}$$

$$k = \frac{2\pi a(v_{mn})}{v}$$

$v_{mn} = 1290 m/s$	Velocity of sound in hydrogen gas
$a = 50AU$	Final radius of solar system
$v = 350 m/s$	Velocity of sound
$k = 1157.901$	k-constant of Bessel function
$n = 0,2,4,6,8, \ldots$	n-constant of wave equation
$t = 1,2,3, \ldots 45$	Time in increments of 100 million years
$\theta = 0 \ldots 2\pi$	Radians
$r = 1,2,3, \ldots 50$	Varying distance from the sun
J_n	Bessel function with respect to r
$\sin n\theta$	Sine with respect to n and theta
$\cos n\theta$	Cosine with respect to n and theta
$COS\ kvt$	Cosine with respect to k, v, and t
P_1	$(J_n)(\sin n\theta)(\cos kvt)$
P_2	$(J_n)(\cos n\theta)(\cos kvt)$

Fig1. The table of variables and constants used in the equations.

RESULTS

I substituted in some initial values for the constant value of the wave equation, time, and theta and using the equation I was able to generate a wave on a graph. Then I looked at the nodal points of the wave and compared the distances from the center to the nodal points to the actual distances of the planets from the sun. Next, I repeated the process using different values for the constant "n" and time. I wanted to keep theta at a constant zero because it is easier to visualize a 2-dimensional graph when the theta is zero rather than any other number. I've included several waves generated by the many different values I tried on one graph to clearly show a difference that is generated in the pressure wave due to just two values. The y-axis of the graph is pressure and the x-axis of the graph is the distance. The x-axis is scaled in such a fashion so that it only extends to 10 AU because the only planets visible to the naked eye are about 10 AU from the sun. I've also included a table of the planets and their distance from the sun in astronomical units for comparison.

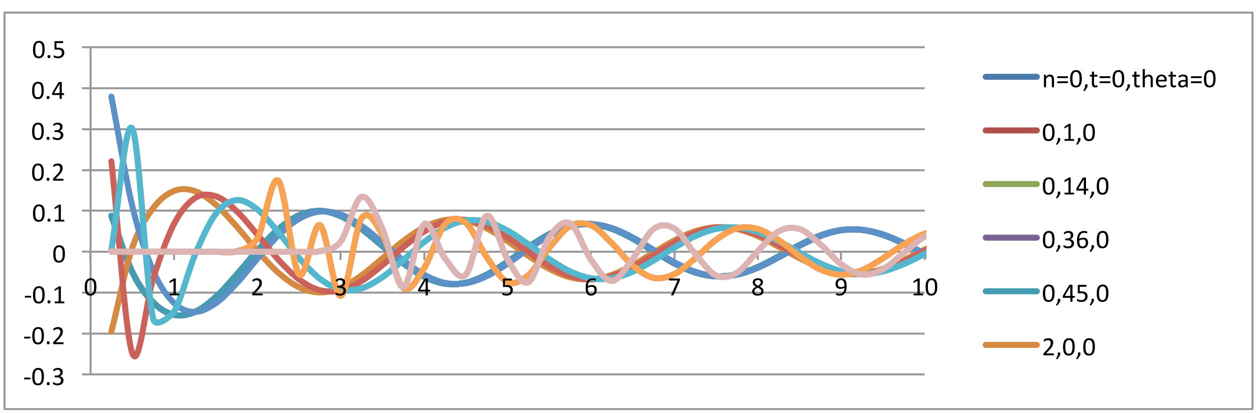

Fig2. The figure above shows the Pressure Wave Amplitudes as a function of Radius.

Mercury	0.39
Venus	0.72
Earth	1
Mars	1.52
Jupiter	5.20
Saturn	9.54

Fig 3. The table above lists the Planetary Orbital Radii (A.U.).

CONCLUSION

I began my research with several books and programs. After weeks of selecting relevant information from the books and picking the correct programs to use, I finally obtained one equation that looked promising. After solving and simplifying the equation for days, I finally came upon a set of variables that I could change to model the evolution of the Solar System. After fine tuning the variables and equation, I ended up with a graph of several waves that depict a rapid, propagating pressure wave emitting from the sun with several nodal points.

I predicted that those nodal points represent the locations of the planets from the sun, and so far the data is too volatile to suggest anything that goes against my prediction. I was able to verify that pressure is affected by several variables and the results suggest that pressure is very dependent on those variables. A slight change in the numbers I chose had a significant impact on the pressure wave. The results show that there is a limit to the numbers I could choose for the constant "n" because as that constant approaches large numbers like 50 and 74, the pressure wave starts propagating much later, around 3 AU. This phenomenon is not possible because four planets are supposed to form within that 3 AU. The results show that pressure is clearly dependent on the varying distances of planets from the sun, the constant "n" of the wave equation, and time.

Because the pressure waves are propagating rapidly throughout the graph, I can conclude that keeping a constant density throughout equation does not yield accurate results because very few nodal points are close to the actual distances of the planets from the sun. I tried to keep density constant throughout my propagating wave equation; however, pressure is too dependent upon density for there to be good results. Several authors kept density constant just to view the effects of radial distances and time on the pressure wave and I followed their models, adding the constant of the wave equation as an extra variable. Like other authors, I was able to view propagating pressure waves. I accomplished that task using my research; however, there is still a lot I could do to continue my research.

The main step forward for my research is to include a varying density into the equation. The varying density would alter the Bessel function in the equation: it would introduce other variables like the "m" constant of the Bessel function, the "k" constant of the Bessel function, and create a varying "v subscript mn" constant. Adding a varying density should generate waves that are much less dynamic than the waves generated in my results now, thus allowing the nodal points to represent actual distances of the planets from the sun. My research was successful as a first step into this project, creating many ways for future candidates to apply this research into any project that requires ascertaining the distances of the planets from the sun.

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